2-Token Dynamic Balancer AMM

The Balancer AMM can be thought of as being an extension of the constant product AMM (defined by the invariant $xy = L^2$), employing exponential weights $\{w_1, w_2\}$ on the reserve variables (x, y), with the invariant equation thus given by:

$$x^{w_1} y^{w_2} = L \tag{1}$$

with the constraint that

$$w_1 + w_2 = 1 \tag{2}$$

In this sense, the constant product AMM can be thought of as the particular case of $w_1 = w_2 = 1/2$.

The instantaneous spot price p is given by the (absolute value of) the slope of the tangent line dy/dx, which we can compute as follows:

$$\frac{d}{dx}\Big(x^{w_1} y^{w_2} = L\Big) \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{w_1}{w_2}\frac{y}{x} \tag{3}$$

Thus, the price is given by

$$p = \frac{w_1}{w_2}\frac{y}{x} \tag{4}$$

When a swap occurs, the token reserves are changed by amounts $(\Delta x, \Delta y)$, one of which obviously being negative. The invariant equation must hold, and so we can say $(x + \Delta x)^{w_1}(y + \Delta y)^{w_2} = L$. Thus, depending on which quantity is given, we can solve for the other:

$$\text{(if } \Delta x \text{ given)} \quad \Delta y = \Big(L/(x + \Delta x)^{w_1}\Big)^{1/w_2} - y \tag{5}$$

$$\text{(if } \Delta y \text{ given)} \quad \Delta x = \Big(L/(y + \Delta y)^{w_2}\Big)^{1/w_1} - x \tag{6}$$

We can also substitute expression (1) in for L, and rewrite (5)-(6) as the following:

$$\text{(if } \Delta x \text{ given)} \quad \Delta y = \left[\left(\frac{x}{x + \Delta x}\right)^{\frac{w_1}{w_2}} - 1\right] y \tag{7}$$

$$\text{(if } \Delta y \text{ given)} \quad \Delta x = \left[\left(\frac{y}{y + \Delta y}\right)^{\frac{w_2}{w_1}} - 1\right] x \tag{8}$$

One can check that in the constant product case ($w_1{=}w_2{=}1/2$), the expressions in (7)-(8) reproduce the familiar formulas for Δx and Δy.

Moreover, in the constant product case, we can use the constant L and price p to solve for x and y, giving us the expressions $x = L/\sqrt{p}$ and $y = L\sqrt{p}$. Similarly, with equations (1) and (4), we can solve for x and y in terms of p once again, although it is slightly more complicated than before:

$$x = L(w_1/w_2)^{w_2}/p^{w_2} \tag{9}$$

$$y = L(w_2/w_1)^{w_1} \cdot p^{w_1} \tag{10}$$

One can easily check that the pair of equations (7) and (8) will reproduce equations (1) and (4).

A user may wish to do a swap with a preset limit price. To do this, we need to determine the incoming token amount that moves the pool from its current price p to some target price p'. For example, if $p > p'$, then we must pay an amount Δx into the pool and receive an amount Δy out of the pool, resulting in a new price p'. We can write down equation (4) in this case, and substitute equation (7) in the numerator:

$$p' = \frac{w_1}{w_2}\frac{(y + \Delta y)}{(x + \Delta x} = \frac{w_1}{w_2}\frac{\big(\frac{x}{x+\Delta x}\big)^{\frac{w_1}{w_2}} y}{(x + \Delta x)} \tag{11}$$

Using the fact that $w_1{+}w_2{=}1$, we can then simplify and rearrange equation (11) into the following:

$$(x + \Delta x) = x\left(\frac{w_1}{w_2}\frac{y}{x}\frac{1}{p'}\right)^{w_2} \tag{12}$$

Recognizing the expression inside the parentheses from equation (4), we can simplify and solve for Δx:

$$\Delta x = \Big[(p/p')^{w_2}\text{-}1\Big] x \tag{13}$$

Similarly, if we have $p' > p$, then we seek an amount Δy to pay into the pool. The analogous calculation to equation (11) would be

$$p' = \frac{w_1}{w_2}\frac{(y + \Delta y)}{(x + \Delta x} = \frac{w_1}{w_2}\frac{(y + \Delta y)}{\big(\frac{y}{y+\Delta y}\big)^{\frac{w_2}{w_1}} x} \tag{14}$$

After much simplification, this can be solved for Δy:

$$\Delta y = \Big[(p'/p)^{w_1}\text{-}1\Big] y \tag{15}$$

Formulas (13) and (15) thus allow us to move the market to a pre-selected price in either direction.

Dynamic Weights

In this framework, our specific goal will be to allow for liquidity providers to add liquidity at arbitrary ratios without affecting the current spot price. In particular, suppose that we have a current price p, and two arbitrary (but positive) quantities $(\Delta x, \Delta y)$ are to be added to the reserves. In order to maintain the price, we need the new weights (w_1', w_2') to be updated so that they now solve the equation:

$$p = \frac{w_1'}{w_2'}\frac{(y + \Delta y)}{(x + \Delta x)} \tag{16}$$

By substituting in $w_2' = 1 - w_1'$, we can solve this equation for w_1', and then subsequently for w_2'. The result is given by the following:

$$w_1' = \frac{p(x + \Delta x)}{p(x + \Delta x) + (y + \Delta y)} \tag{17}$$

$$w_2' = \frac{(y + \Delta y)}{p(x + \Delta x) + (y + \Delta y)} \tag{18}$$

One easily checks that $w_1' + w_2' = 1$ still holds.

SUMMARY

- The pool invariant is given by $\boxed{x^{w_1}y^{w_2} = L}$ such that $\boxed{w_1 + w_2 = 1}$

- The instantaneous price is given by $\boxed{p = \frac{w_1}{w_2}\frac{y}{x}}$

- Swaps are computed by:

  (if Δx given) $\boxed{\Delta y = y\left[\left(x/(x+\Delta x)\right)^{\frac{w_1}{w_2}} \text{-} 1\right]}$

  (if Δy given) $\boxed{\Delta x = x\left[\left(y/(y+\Delta y)\right)^{\frac{w_2}{w_1}} \text{-} 1\right]}$

- To reach a target price p' from a current price p, the required incoming amounts are given by

  (if $p' < p$) $\boxed{\Delta x = x\left[(p/p')^{w_2}\text{-}1\right]}$

  (if $p' > p$) $\boxed{\Delta y = y\left[(p'/p)^{w_1}\text{-}1\right]}$

- When the amounts $(\Delta x, \Delta y)$ are *injected*, the weights (w_1, w_2) are updated to (w_1', w_2') by:

$$\boxed{w_1' = \frac{p(x+\Delta x)}{p(x+\Delta x) + (y+\Delta y)}}$$

$$\boxed{w_2' = \frac{(y+\Delta y)}{p(x+\Delta x) + (y+\Delta y)}}$$